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centrum

AFDELING MATHEMATISCHE STATISTIEK SW 10/71 JUNE

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ON R. VON MISES' CONDITION FOR THE DOMAIN OF ATTRACTION OF EXP($-e^{-x}$)

Prepublication

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

Abstract

On R. von Mises' condition for the domain of attraction of $\exp(-e^{-x})$

There exist well-known necessary and sufficient conditions for a distribution function to belong to the domain of attraction of the double exponential distribution Λ. For practical purposes a simple sufficient condition due to Von Mises is very useful. It is shown that each distribution function F in the domain of attraction of Λ is close to some distribution function satisfying Von Mises' condition.

This report is an improved version of report SW 8/71.

On R. von Mises' condition for the domain of attraction of $\exp(-e^{-x})$.*)

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Suppose X_1, X_2, X_3, ... are independent real-valued random variables with common distribution function F. We say that F is in the domain of attraction of the double exponential distribution (notation $F \in D(\Lambda)$; $\Lambda(x) = \exp(-e^{-x})$) if there exist two sequences of real constants $\{b_n\}$ and $\{a_n\}$ (with $a_n > 0$ for n = 1, 2, ...) such that for all real x

$$(1) \qquad \lim_{n\to\infty} P\left\{\frac{\max(X_1,X_2,\ldots,X_n)-b_n}{a_n} \leq x\right\} = \exp(-e^{-x}).$$

Necessary and sufficient conditions for $F \in D(\Lambda)$ are well-known ([1] and [2]) but rather intricate. The following relatively simple criterion is due to R. von Mises ([4] p. 285). It is convenient for the formulation of the theorem to use the symbol x_0 for the upper bound of X_1 defined by

$$x_0(F) = \sup\{x \mid F(x) < 1\}.$$

<u>Theorem 1</u>. Suppose F(x) is a distribution function with a density f(x) which is positive and differentiable on a left neighbourhood of x_0. If

$$(2) \qquad \lim_{x\uparrow x_0} \frac{d}{dx}\left(\frac{1-F(x)}{f(x)}\right) = 0,$$

then $F \in D(\Lambda)$.

A distribution function F satisfying (2) will be called a <u>Von Mises function</u>.

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*) Report SW 10/71, Afdeling Mathematische Statistiek, Mathematisch Centrum, Amsterdam.

We shall prove

<u>Theorem 2</u> A distribution function F lies in the domain of attraction of Λ if and only if there exists a Von Mises function F_* such that

(3) $$\lim_{x \uparrow x_0} \frac{1-F(x)}{1-F_*(x)} = 1.$$

For the proof we need three lemma's.

<u>Lemma 1</u> Let F and G be distribution functions and let $a_n > 0$ and b_n be real constants such that

(4) $$\lim_{n\to\infty} G^n(a_n x+b_n) = \Lambda(x).$$

If

(5) $$\lim_{x \uparrow x_0(G)} \frac{1-F(x)}{1-G(x)} = 1$$

then

(6) $$x_0(F) = x_0(G)$$

and

(7) $$\lim_{n\to\infty} F^n(a_n x+b_n) = \Lambda(x).$$

<u>Proof of lemma</u> Since $0 \leq \Lambda(x) < 1$ for all real x relation (4) implies

(8) $$\lim_{n\to\infty} 1 - G(a_n x+b_n) = 0$$

and hence

(9) $a_n x + b_n$ converges to x_0 from the left for $n \to \infty$,

which together with (5) implies (6).

The relation (8) implies that

$$n\{1-G(a_n x+b_n)\} \sim \log G^n(a_n x+b_n) \sim \log \Lambda(x) \quad \text{for } n \to \infty.$$

These asymptotic relations also hold for F instead of G because of (9) and (5). This proves (7). Q.E.D.

In order to increase the differentiability of F we define the sequence of distribution functions F_0, F_1, F_2, ... by

$$F_0(x) = F(x)$$

$$1 - F_{n+1}(x) = \min\{1, \int_x^{x_0} (1-F_n(t))dt\} \qquad \text{for } x < x_0.$$

Lemma 2 If $F \in D(\Lambda)$, then $F_1 \in D(\Lambda)$. In particular the sequence F_n is well defined.

Proof See De Haan [2] lemma 2.5.1 or [3] lemma 6.

Lemma 3 If $F \in D(\Lambda)$, then

$$(10) \qquad \lim_{x \uparrow x_0} \frac{\{1-F(x)\}.\{1-F_2(x)\}}{\{1-F_1(x)\}^2} = 1.$$

Proof See De Haan [2] th. 2.5.2 or [3] th. 10.

Note that (10) is the integral form of (2).

Corollary If $F \in D(\Lambda)$, then for n = 1, 2, ...

$$(11) \qquad \lim_{x \uparrow x_0} \frac{\{1-F_{n-1}(x)\}\ \{1-F_{n+1}(x)\}}{\{1-F_n(x)\}^2} = 1.$$

Proof of theorem 2 The if statement is a trivial result of lemma 1. Now suppose $F \in D(\Lambda)$.

Define the function U_* by

$$U_*(x) = U_3^4(x)\ U_4^{-3}(x)$$

where $U_n(x) = 1-F_n(x)$. Then $U\ (x)$ is twice differentiable in a left neighbourhood of x_0 and

$$(12)\qquad \frac{d}{dx}\log U_* = -4\,\frac{U_2}{U_3} + 3\,\frac{U_3}{U_4} = \frac{3-4\ U_2U_3^{-2}U_4}{U_4U_3^{-1}} .$$

Consider

$$\frac{U_4U_3^{-1}}{3-4\ U_2U_3^{-2}U_4} = \frac{U_*}{\frac{d}{dx}U_*} .$$

By (11) the denominator is asymptotic to -1 for $x \uparrow x_0$ and both $\frac{d}{dx}U_4U_3^{-1}$ and $U_4U_3^{-1}\frac{d}{dx}(3-4\ U_2U_3^{-2}U_4)$ vanish for $x \uparrow x_0$. Hence

$$(13)\qquad \lim_{x\uparrow x_0} \frac{d}{dx}\left(\frac{U_*(x)}{U_*'(x)}\right) = 0.$$

Observe that

$$U_0 = \frac{U_0U_2}{U_1^2}\cdot\left(\frac{U_1U_3}{U_2^2}\right)^2\cdot\left(\frac{U_2U_4}{U_3^2}\right)^3\cdot U_* .$$

Hence by (11) we obtain

$$(14)\qquad \lim_{x\uparrow x_0}\frac{U_0(x)}{U_*(x)} = 1.$$

Then $\lim_{x\uparrow x_0} U_*(x) = 0$, and since by (12) U_* is decreasing on a left neighbourhood of x_0, there exists a twice differentiable distribution function $F_*(x)$ which coincides with $1 - U_*(x)$ on a left neighbourhood of x_0. F_* is a Von Mises function by (13) and satisfies (2) by (14). Q.E.D.

Remarks Lemma 1 is a particular case of a theorem due to Resnick [5] (th. 2.3). Lemma 1 implies that for the convergence of the distribution functions F^n and F_*^n the same norming constants $a_n > 0$ and b_n may be used.

Corollary If $F \in D(\Lambda)$, there exist a positive function c satisfying $\lim_{x\uparrow x_0} c(x) = 1$ and a positive differentiable function ϕ satisfying

$\phi(x_0) = 0$ if x_0 is finite and $\lim_{x \uparrow x_0} \phi'(x) = 0$ such that

$$1 - F(x) = c(x).\ \exp\{-\int_{-\infty}^{x} \frac{dt}{\phi(t)}\} \qquad \text{for } x < x_0.$$

This improves the representation theorem 2.5.3 in De Haan [2].

Proof Set $\phi(x) = \dfrac{1-F_*(x)}{F_*'(x)}$ in a left neighbourhood of x_0.

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